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2007 J. Phys. A: Math. Theor. 40 11129

(http://iopscience.iop.org/1751-8121/40/36/013)

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) 11129–11149 doi:10.1088/1751-8113/40/36/013

Berry phases for 3D Hartree-type equations with a quadratic potential and a uniform magnetic field

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Received 30 October 2006, in final form 20 July 2007
Published 21 August 2007
Online at stacks.iop.org/JPhysA/40/11129

Abstract
A countable set of asymptotic space-localized solutions is constructed for a 3D Hartree-type equation with a quadratic potential by the complex germ method in the adiabatic approximation. The asymptotic parameter is $1/T$, where $T \gg 1$ is the adiabatic evolution time. A generalization of the Berry phase of the linear Schrödinger equation is formulated for the Hartree-type equation. For the solutions constructed, the Berry phases are found in an explicit form.

PACS numbers: 02.30.Jr, 03.65.Sq

1. Introduction

In the course of adiabatic evolution, a quantum system returns to its initial state and the wavefunction gains only a phase factor. Berry has revealed [1] that the total phase contains, along with the dynamic part known from the Born–Fock adiabatic theorem [2–4], a geometric additive component (geometric phase (GP)). This summand in the total phase is known as the adiabatic phase or the Berry phase. The adiabatic phase is closely connected with the Floquet problem for systems of differential equations with periodic coefficients. In quantum mechanics, geometric phases are well investigated for the linear Schrödinger equation [2, 5, 6]. In classical mechanics, a Hannay angle is introduced for nearly integrable Hamiltonian systems with adiabatically varying parameters. The Hannay angle is related to the Berry phase [7] if the Hamiltonian system corresponds to the quantum system under consideration. The Hannay angle is defined as an additional term to the 'angle' variable when the Hamiltonian system is described in terms of 'action–angle' variables [8, 9].

Geometric phases are observable and they show up in various physical phenomena [10–14]. It is believed that in quantum calculations [15, 16], intensely developing at present, the Berry phase might be used in some types of quantum gate, the so-called geometric gates [17]. The geometric gates offer some advantages over the conventional (non-geometric) phase

gates owing to their greater fault tolerance. An example of practical realization of geometric gates based on nuclear magnetic resonance is given in [18]. To build hardware for a quantum computer, systems of cooled ions in a Paul trap can be used [16]. In such a system, each ion carries a qubit, and a logic operation (gate) is governed by an external electromagnetic field created by laser radiation, a magnetic field source, etc. It should be noted that a Paul trap can be described by the potential of a harmonic oscillator [11, 16]. One should also take into account the collective coupling of ions, as the ion states depend not only on the external field, but also on the collective behavior of the ions [16]. A consideration of the coupling between parts of a system naturally leads to nonlinear models. An example of such a system might be the Bose–Einstein condensate (BEC) whose models use the Gross–Pitaevskii equation [19, 20]. This is an argument in favor of studying models of this type bearing in mind their applications.

Here, we continue our investigations [21, 22] of the geometric phases for the one-dimensional Hartree-type equation. The goal of the present work is to find the Berry phase for a many-dimensional Hartree-type equation of the following form:

$$\{-i\hbar\partial_t + \widehat{\mathcal{H}}_\varkappa(R(t), \Psi)\}\Psi = 0, \tag{1.1}$$

$$\widehat{\mathcal{H}}_\varkappa(R(t), \Psi) = \widehat{\mathcal{H}}(R(t)) + \varkappa\widehat{V}(\Psi), \tag{1.2}$$

$$\widehat{\mathcal{H}}(R(t)) = \frac{1}{2m(t)}\left(\hat{\vec{p}} - \frac{e}{c}\vec{A}(\vec{x}, t)\right)^2 + \frac{\rho(t)(\langle\vec{x}, \hat{\vec{p}}\rangle + \langle\hat{\vec{p}}, \vec{x}\rangle)}{2} + \frac{k(t)\vec{x}^2}{2}, \tag{1.3}$$

$$\widehat{V}(\Psi) = \int_{\mathbb{R}^3} \mathrm{d}\vec{y}\, W(\vec{x}, \vec{y})|\Psi(\vec{y})|^2, \tag{1.4}$$

$$\vec{A}(\vec{x}, t) = \frac{1}{2}[\vec{H}(t), \vec{x}]. \tag{1.5}$$

Here, $\vec{x} \in \mathbb{R}^3$, $\hat{\vec{p}} = -i\hbar\partial/\partial\vec{x}$, $t \in \mathbb{R}^1$, $\partial_t = \partial/\partial t$; $m(t), k(t)$ and $\rho(t)$ are the time-dependent parameters of the system, $\vec{H}(t) = \{H_1(t), H_2(t), H_3(t)\}$ is the external magnetic field; $\varkappa$ is the nonlinearity parameter; $R(t)$=$(m(t), k(t), \rho(t), \vec{H}(t))$; $\langle\vec{a}, \vec{b}\rangle$ and $[\vec{a}, \vec{b}]$ are, respectively, the scalar and the vector product in $\mathbb{R}^3$.

The operator $\widehat{\mathcal{H}}(R(t))$ consists of summands having a well-defined physical meaning. The operator $\widehat{\mathcal{H}}(R(t))$ can be considered the Hamiltonian of a quantum system in the external field of a generalized harmonic oscillator and the homogeneous magnetic field specified by a vector potential $\vec{A}(\vec{x}, t)$. The presence of a magnetic field reduces the symmetry of a quantum system, removing its degeneration, which, in turn, results in some physical effects, such as the normal Zeeman effect [23], etc. The summand $k(t)x^2/2$ can be considered a model of a spherical Paul trap for cooled ions [11]. The operator $(\langle\vec{x}, \hat{\vec{p}}\rangle + \langle\hat{\vec{p}}, \vec{x}\rangle)$ is essential in the study of the squeezed states of a system in the field of a harmonic oscillator and, thus, the nontrivial Berry phase, in various physical problems (see [24, 25] for details). The nonlinear term (1.4) is responsible for the coupling of the system particles in the mean-field approximation (see, e.g., [4]). The coupling potential $W(\vec{x}, \vec{y})$ is assumed to be a smooth function of its arguments, which possesses the property of permutable and coordinated invariance, i.e. $W(\vec{x}, \vec{y}) = W(\vec{y}, \vec{x})$, $W(\vec{x}, \vec{y}) = W(\vec{x} + \vec{q}, \vec{y} + \vec{q})$.

To construct the leading term of a solution asymptotic in a small parameter $\hbar$, the quadratic approximation of the coupling potential $W(\vec{x}, \vec{y}) = 1/2[a(t)\vec{x}^2 + 2b(t)\langle\vec{x}, \vec{y}\rangle + c(t)\vec{y}^2]$ is shown to be sufficient in the class of functions localized in the neighborhood of a phase trajectory (see [26, 27]). We consider the parameters $a(t)$, $c(t)$ and $b(t)$ to be time dependent for mathematical generality. Symmetry properties of the potential $W(\vec{x}, \vec{y})$ result in $a(t) = c(t) = -b(t)$.

We construct the solutions of equation (1.1) in the adiabatic approximation and obtain the Berry phases for a system of coherent ions subject to their mutual interaction and periodical

dependence of the nonlinear Hamiltonian $\widehat{\mathcal{H}}_\varkappa(R(t), \Psi)$ (1.2). Hereinafter, we imply by $R(t)$ a set of parameters: $R(t) = (m(t), k(t), \rho(t), \vec{H}(t), a(t), b(t), c(t))$.

The study of the geometric phase for charged particles in harmonic potential traps in the presence of a magnetic field is a problem of widespread interest [11, 28–30]. Nonlinear models for systems of interacting charged particles are discussed in the context of the elaboration of the element base for quantum computers. In [31], the quantum fidelity decay in a system of charged particles was studied within the framework of the mean-field approximation. It has been found that an ensemble of interacting charged particles is more stable to perturbations than a single-particle system and this can bring into existence qubits based on coherent ensembles of interacting particles. Thus, the mathematical modeling of geometric gates for such qubits becomes a topical problem, and our research makes a contribution to the study of these mathematical models.

In the development of analytic methods in the geometric phase theory, the key point is the integrability of nonlinear partial differential equations with variable coefficients. This problem requires particular mathematical ideas and constructions. The well-known inverse scattering transform method [32, 33] is applied mainly to $(1+1)D$ and $(1+2)D$ nonlinear equations with constant coefficients, and only soliton solutions can be constructed in an explicit form. The symmetry analysis [34, 35] allows one to study systems possessing high symmetry, but the evaluation of the degree of symmetry is inconvenient if the equation contains nonlocal terms. Nonlinear problems of the above class could be solved efficiently by the method developed in [26, 27], where asymptotic solutions (in some cases, exact solutions) are constructed for a nonlinear Hartree-type equation which is a Gross–Pitaevskii equation with a nonlocal nonlinearity.

This paper is organized as follows. In section 2, necessary designations and definitions are introduced for the Berry phase. In section 3, a method of seeking an exact solution for a Hartree-type equation with a quadratic Hamiltonian is briefly described. Section 4 presents the solution of the spectral problem for an instantaneous Hartree-type operator. The solution is used to extract the dynamic phase from the overall phase. In section 5, the solutions of the Hartree-type equation are constructed in the adiabatic approximation and the corresponding Berry phases are found. In the conclusion, the results and related problems are discussed.

2. The Berry phase for the nonlinear equation

In this work, we find the Berry phase using the approach developed in [36, 37] for linear quantum equations. This approach is based on seeking an exact (or approximate) solution of the Cauchy problem for equation (1.1),

$$\Psi|_{t=0} = \psi_\nu(\vec{x}, R(0)), \tag{2.1}$$

which then is expanded in the adiabatic parameter. Here, the functions $\psi_\nu(\vec{x}, R(0))$ are determined by the spectral problem for the instantaneous Hartree-type operator

$$\widehat{\mathcal{H}}_\varkappa(R(t))\psi_\nu(\vec{x}, R(t)) = E_\nu(R(t))\psi_\nu(\vec{x}, R(t)). \tag{2.2}$$

Assume that the parameters $R(t)$ are T-periodic and slowly vary with time[3]. Following the adiabatic theorem in (linear) quantum mechanics [4], we seek a solution of the Cauchy

[3] A system is assumed to be adiabatic if the following condition is fulfilled:

$$\max_{i=\overline{1,n}} \dot{R}_i \frac{T}{R_i} \ll 1, \tag{2.3}$$

where R_i are the parameters of the Hamiltonian (see, e.g., [38]).

problem (1.1), (2.1) in the form

$$\Psi(\vec{x}, t) = \exp[\mathrm{i}\phi_\nu(t)]\psi_\nu(\vec{x}, R(t)) + O\left(\frac{1}{T}\right). \tag{2.4}$$

Then for $t = T$, taking into account that $R(T) = R(0)$, we have

$$\Psi(\vec{x}, T) = \exp[\mathrm{i}\phi_\nu(T)]\Psi(\vec{x}, 0) + O\left(\frac{1}{T}\right). \tag{2.5}$$

Let us rewrite the phase $\phi_\nu(T)$ as

$$\phi_\nu(T) = \delta_\nu(T) + \gamma_\nu(T), \tag{2.6}$$

where $\delta_\nu(T)$ is the dynamic phase, which is defined by the relation

$$\delta_\nu(T) = -\frac{1}{\hbar}\int_0^T E_\nu(R(t))\,\mathrm{d}t. \tag{2.7}$$

Following [1] (see also [5, 6, 22]), we call the phase $\gamma_\nu(T)$ the adiabatic Berry phase or geometric phase for equation (1.1). The Berry phase for the linear Schrödinger equation ($\varkappa = 0$ in (1.2)) is determined by

$$\gamma_\nu^{\mathrm{lin}}(T) = \mathrm{i}\int_0^T \langle\psi_\nu(\vec{x}, R(t))|\dot{\psi}_\nu(\vec{x}, R(t))\rangle\,\mathrm{d}t. \tag{2.8}$$

Formula (2.8) is equivalent to (2.6) in the linear case and requires an additional substantiation for nonlinear equations.

3. Method of semiclassically concentrated functions

We consider exact solutions for equation (1.1), following [26, 27]. From these solutions we find solutions of the form (2.4) and thus obtain the Berry phase.

Define the mean value for a linear operator $\widehat{A}$ in a state $\Psi(t)$ as

$$\begin{aligned}\langle\widehat{A}(t)\rangle = A_\Psi(t,\hbar) &= \frac{1}{\|\Psi(t)\|^2}\langle\Psi(t)|\widehat{A}|\Psi(t)\rangle \\ &= \frac{1}{\|\Psi(t)\|^2}\int_{\mathbb{R}^3}\mathrm{d}\vec{x}\,\Psi^*(\vec{x}, t, \hbar)\widehat{A}(t)\Psi(\vec{x}, t, \hbar),\end{aligned} \tag{3.1}$$

where $\|\Psi(t)\|^2 = \langle\Psi(t)|\Psi(t)\rangle$. For the solutions $\Psi(t)$ of equation (1.1) we have

$$\frac{\mathrm{d}\langle\widehat{A}(t)\rangle}{\mathrm{d}t} = \left\langle\frac{\partial\hat{A}(t)}{\partial t}\right\rangle + \frac{\mathrm{i}}{\hbar}\langle[\widehat{\mathcal{H}}_\varkappa(t, \Psi(t)), \widehat{A}(t)]_-\rangle, \tag{3.2}$$

where $[\hat{A}, \hat{B}]_- = \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator of linear operators $\hat{A}$ and $\hat{B}$.

As in the linear case, we call equation (3.2) an Ehrenfest equation. The norm of a solution of equation (1.1) does not depend on time, i.e.

$$\|\Psi(\vec{x}, t)\|^2 = \|\Psi(\vec{x}, 0)\|^2 = \|\Psi\|^2.$$

Then it is convenient to change the constant $\varkappa$ by a constant $\tilde{\varkappa} = \varkappa\|\Psi\|^2$.

Let us write the Ehrenfest equation for the mean values of the operator $\hat{\mathfrak{g}}$:

$$\begin{aligned}&\hat{\mathfrak{g}} = \left(\hat{z}_j, \frac{1}{2}(\Delta\hat{z}_k\Delta\hat{z}_l + \Delta\hat{z}_l\Delta\hat{z}_k);\ j, k, l = \overline{1,6}\right)^{\mathsf{T}};\\ &\hat{z}_m = \hat{p}_m, \qquad \hat{z}_{m+3} = x_m, \qquad m = \overline{1,3},\end{aligned} \tag{3.3}$$

where $\Delta\hat{z}_k = \hat{z}_k - z_{k\Psi}(t,\hbar)$, $z_{k\Psi}(t,\hbar) = \langle\hat{z}_k\rangle$. As a consequence, we obtain for the first- and second-order moments

$$\begin{cases} \dot{z}_\Psi = J\mathcal{H}_z(\mathfrak{g}_\Psi, R(t)), \\ \dot{\Delta}_{2\Psi} = J\mathcal{H}_{zz}(R(t))\Delta_{2\Psi} - \Delta_{2\Psi}\mathcal{H}_{zz}(R(t))J, \quad \Delta_{2\Psi}^{\intercal} = \Delta_{2\Psi}, \quad J = \begin{pmatrix} 0 & -\mathbb{I}_{3\times3} \\ \mathbb{I}_{3\times3} & 0 \end{pmatrix}, \end{cases} \tag{3.4}$$

$$\mathcal{H}_z(\mathfrak{g}_\Psi, R(t)) = \begin{pmatrix} \dfrac{\vec{p}_\Psi}{m(t)} + \rho(t)\vec{x}_\Psi - \dfrac{e}{2m(t)c}[\vec{H}(t), \vec{x}_\Psi] \\ k_1(t)\vec{x}_\Psi + \rho(t)\vec{p}_\Psi + \frac{e}{2m(t)c}[\vec{H}(t), \vec{p}_\Psi] + \frac{e^2}{4m(t)c^2}[\vec{H}(t), [\vec{x}_\Psi, \vec{H}(t)]] \end{pmatrix}, \tag{3.5}$$

$$\mathcal{H}_{zz}(R(t)) = \begin{pmatrix} \mathcal{H}_{pp}(R(t)) & \mathcal{H}_{px}(R(t)) \\ \mathcal{H}_{px}^{\intercal}(R(t)) & \mathcal{H}_{xx}(R(t)) \end{pmatrix}, \tag{3.6}$$

$$\mathcal{H}_{pp}(R(t)) = \frac{1}{2m(t)}\mathbb{I}_{3\times3}, \qquad \mathcal{H}_{px}(R(t)) = \frac{e}{2m(t)c}\begin{pmatrix} \frac{2m(t)\rho(t)c}{e} & H_3(t) & -H_2(t) \\ -H_3(t) & \frac{2m(t)\rho(t)c}{e} & H_1(t) \\ H_2(t) & -H_1(t) & \frac{2m(t)\rho(t)c}{e} \end{pmatrix}, \tag{3.7}$$

$$\mathcal{H}_{xx}(R(t)) = \frac{e^2}{4m(t)c^2} \times \begin{pmatrix} \vec{H}^2(t) - H_1^2(t) + \frac{4m(t)\tilde{k}(t)c^2}{e^2} & -H_1(t)H_2(t) & -H_1(t)H_3(t) \\ -H_2(t)H_1(t) & \vec{H}^2(t) - H_2^2(t) + \frac{4m(t)\tilde{k}(t)c^2}{e^2} & -H_2(t)H_3(t) \\ -H_3(t)H_1(t) & -H_2(t)H_1(t) & \vec{H}^2(t) - H_3^2(t) + \frac{4m(t)\tilde{k}(t)c^2}{e^2} \end{pmatrix}. \tag{3.8}$$

Here J is a standard symplectic matrix, $\mathbb{I}_{n\times n}$ is an identity matrix of dimension n, $B^{\intercal}$ is the transpose to the matrix B; $k_1(t) = k(t) + \tilde{\varkappa}(a(t) + b(t))$, $\tilde{k}(t) = k(t) + \tilde{\varkappa}a(t)$.

The matrix of the second centred moment $\Delta_{2\Psi}$ has the form

$$\Delta_{2\Psi}(t,\hbar) = \begin{pmatrix} \sigma_{pp}^{\Psi}(t,\hbar) & \sigma_{px}^{\Psi}(t,\hbar) \\ \sigma_{xp}^{\Psi}(t,\hbar) & \sigma_{xx}^{\Psi}(t,\hbar) \end{pmatrix}, \tag{3.9}$$

where

$$\begin{aligned} \sigma_{pp}^{\Psi}(t,\hbar) &= \left\|\sigma_{p_kp_l}^{\Psi}(t,\hbar)\right\|_{3\times3} = \|\langle\Delta\hat{p}_k\Delta\hat{p}_l\rangle)\|_{3\times3}, \\ \sigma_{xx}^{\Psi}(t,\hbar) &= \left\|\sigma_{x_kx_l}^{\Psi}(t,\hbar)\right\|_{3\times3} = \|\langle\Delta x_k\Delta x_l\rangle\|_{3\times3}, \qquad k,l = \overline{1,3} \\ \sigma_{xp}^{\Psi}(t,\hbar) &= \left\|\sigma_{x_kp_l}^{\Psi}(t,\hbar)\right\|_{3\times3} = \left\|\tfrac{1}{2}\langle\Delta x_k\Delta\hat{p}_l + \Delta\hat{p}_l\Delta x_k\rangle\right\|_{3\times3}. \end{aligned}$$

We call the system of equations (3.4) the second-order *Hamilton–Ehrenfest system* (HES) corresponding to equation (1.1).

The matrix of the second centred moments[4] Δ_2 can be rewritten as

$$\Delta_2(t) = A(t)\Delta_2(0)A^{+}(t), \tag{3.10}$$

where $A(t)$ is the fundamental matrix of the system in variations, i.e.

$$\dot{A} = J\mathcal{H}_{zz}(R(t))A, \qquad A(0) = \mathbb{I}_{6\times6}. \tag{3.11}$$

[4] The subscript Ψ can be omitted.

We denote the general solution of the system (3.4) by $\mathfrak{g}(t,\mathfrak{C}) = (Z(t,\mathfrak{C}), \Delta_2(t,\mathfrak{C}))$, where $Z(t,\mathfrak{C}) = (\vec{P}(t,\mathfrak{C}), \vec{X}(t,\mathfrak{C}))$, and $\mathfrak{C}$ is the set of integration constants $\mathfrak{C} = (c_1,\dots,c_N)$, N is the number of linearly independent equations of system (3.4).

Let us seek a solution of equation (1.1) in terms of the anzats

$$\Psi(\vec{x},t,\hbar) = \varphi\left(\frac{\Delta\vec{x}}{\sqrt{\hbar}},t,\sqrt{\hbar}\right)\exp\left[\frac{\mathrm{i}}{\hbar}(S(t,\hbar)+\langle\vec{P}(t,\hbar),\Delta\vec{x}\rangle)\right]. \tag{3.12}$$

Here, the function $\varphi(\vec{\xi},t,\sqrt{\hbar})$ belongs to the Schwartz space $\mathbb{S}$ in the variable $\vec{\xi} = \Delta\vec{x}/\sqrt{\hbar}$ and regularly depends on $\sqrt{\hbar}$ (i.e. can be expanded in a Tailor series of $\sqrt{\hbar}$); $\Delta\vec{x} = \vec{x} - \vec{X}(t,\hbar)$. The real function $S(t,\hbar)$ and the vector function $Z(t,\hbar) = (\vec{P}(t,\hbar), \vec{X}(t,\hbar))$ are to be determined. The Schwartz space is a space of rapidly decreasing indefinitely differentiable functions.

Let us expand the operators entering equation (1.1) in Taylor series of $\Delta\vec{x} = \vec{x} - \vec{x}_\Psi(t,\hbar)$, $\Delta\vec{y} = \vec{y} - \vec{x}_\Psi(t,\hbar)$ and $\Delta\hat{\vec{p}} = \hat{\vec{p}} - \vec{p}_\Psi(t,\hbar)$. Then equation (1.1) takes the form

$$\{-\mathrm{i}\hbar\partial_t + \mathcal{H}(\Psi,t) + \langle\mathfrak{H}_z(\Psi,t),\Delta\hat{z}\rangle + \tfrac{1}{2}\langle\Delta\hat{z},\mathcal{H}_{zz}(t)\Delta\hat{z}\rangle\}\Psi = 0, \tag{3.13}$$

$$\begin{aligned}\mathcal{H}(\Psi,t) = \mathcal{H}(\mathfrak{g}_\Psi(t,\hbar),R(t)) &= \frac{\vec{p}_\Psi^{\,2}(t,\hbar)}{2m(t)} + \rho(t)\langle\vec{x}_\Psi,\vec{p}_\Psi\rangle \\ &+ \frac{k_0(t)\vec{x}_\Psi^{\,2}(t,\hbar)}{2} - \frac{e}{2m(t)c}\langle[\vec{H}(t),\vec{x}_\Psi(t,\hbar)],\vec{p}_\Psi(t,\hbar)\rangle \\ &+ \frac{e^2}{8m(t)c^2}\left[\vec{H}^2(t)\vec{x}_\Psi^{\,2}(t,\hbar) - (\langle\vec{H}(t),\vec{x}_\Psi(t,\hbar)\rangle)^2\right] + \sum_{k=1}^{3}\frac{\tilde{\varkappa}}{2}c(t)\sigma^\Psi_{x_kx_k}(t,\hbar),\end{aligned} \tag{3.14}$$

$$\mathcal{H}_z(\Psi,t) = \mathcal{H}_z(\mathfrak{g}_\Psi(t,\hbar),R(t)), \qquad \mathcal{H}_{zz}(t) = \mathcal{H}_{zz}(R(t)). \tag{3.15}$$

Here $\Delta\hat{z} = (\Delta\hat{\vec{p}},\Delta\vec{x})$, $k_0(t) = k(t) + \tilde{\varkappa}(a(t)+2b(t)+c(t))$; the vector $\mathcal{H}_z(\mathfrak{g}_\Psi,R(t))$ and matrix $\mathcal{H}_{zz}(R(t))$ are given by (3.5), (3.6), respectively.

Let us relate the nonlinear equation (3.13) with the linear equation that is obtained from (3.13) by formal replacement of the mean values of the operators of coordinates, momenta and centred moments of the second order, $\mathfrak{g}_\Psi(t,\hbar)$, by the general solution $\mathfrak{g}(t,\mathfrak{C})$ of the Hamilton–Ehrenfest system (3.4),

$$\begin{aligned}&\left\{-\mathrm{i}\hbar\partial_t + \mathcal{H}(t,\mathfrak{C}) + \langle\mathcal{H}_z(t,\mathfrak{C}),\Delta\hat{z}\rangle + \tfrac{1}{2}\langle\Delta\hat{z},\mathcal{H}_{zz}(t)\Delta\hat{z}\rangle\right\}\Phi = 0, \\ &\mathcal{H}(t,\mathfrak{C}) = \mathcal{H}(\Psi,t)|_{\mathfrak{g}_\Psi(t)\to\mathfrak{g}(t,\mathfrak{C})}, \qquad \mathcal{H}_z(t,\mathfrak{C}) = \mathcal{H}_z(\Psi,t)|_{\mathfrak{g}_\Psi(t)\to\mathfrak{g}(t,\mathfrak{C})}.\end{aligned} \tag{3.16}$$

We call equation (3.16) *the associated linear Schrödinger equation*. We can immediately verify that the function

$$\begin{aligned}\Phi_0(x,t,\mathfrak{C}) = |0,t,\mathfrak{C}\rangle &= N_\hbar\left(\frac{\det C(0)}{\det C(t)}\right)^{1/2} \\ &\times\exp\left\{\frac{\mathrm{i}}{\hbar}\left(S(t,\mathfrak{C}) + \langle\vec{P}(t,\mathfrak{C}),\Delta\vec{x}\rangle + \frac{1}{2}\langle\Delta\vec{x},Q(t)\Delta\vec{x}\rangle\right)\right\},\end{aligned} \tag{3.17}$$

where

$$S(t,\mathfrak{C}) = \int_0^t(\langle\vec{P}(t,\mathfrak{C}),\dot{\vec{X}}(t,\mathfrak{C})\rangle - \mathcal{H}(t,\mathfrak{C}))\,\mathrm{d}t, \tag{3.18}$$

is a solution of equation (3.16). Here $Q(t) = B(t)C^{-1}(t)$; $B(t)$ and $C(t)$ designate the 'momentum' and the 'coordinate' part of the matrix solution of the system in variations (see [39]) corresponding to the linear equation (3.16):

$$\begin{cases}\dot{B} = -\mathcal{H}_{xp}(t)B - \mathcal{H}_{xx}(t)C, \\ \dot{C} = \mathcal{H}_{pp}(t)B + \mathcal{H}_{px}(t)C.\end{cases} \tag{3.19}$$

Let us write a matrix

$$\mathcal{A}(t) = \begin{pmatrix} B(t) \\ C(t) \end{pmatrix}. \tag{3.20}$$

Note that $\mathcal{A}(t) = A(t)\mathcal{A}(0)$, where $A(t)$ is defined by (3.11). The matrix $\mathcal{A}$ can be represented as

$$\mathcal{A} = (a_1, a_2, a_3), \tag{3.21}$$

where $\{a_k\}$ is a set of linearly independent vectors being solutions of the equation

$$\dot{a}_k = J\mathcal{H}_{zz}(t)a_k, \qquad k = \overline{1,3}. \tag{3.22}$$

An operator[5]

$$\hat{a}(t) = N_a\langle a(t), J\Delta\hat{z}\rangle \tag{3.23}$$

is a symmetry operator for equation (3.16), if the vector $a(t)$ is a solution of the system in variations (3.22) [39]. Let $\hat{a}(t)$ and $\hat{b}(t)$ be the symmetry operators corresponding to the two solutions of the system in variations, $a(t)$ and $b(t)$, respectively. Then it is easy to verify that

$$[\hat{a}(t), \hat{b}(t)]_- = -\mathrm{i}\hbar N_a N_b\{a(t), b(t)\} = -\mathrm{i}\hbar N_a N_b\{a(0), b(0)\}. \tag{3.24}$$

The last equality of (3.24) is due to the Hamiltonian form of system (3.22). By braces we designate the skew-scalar product of two vectors $\{a, b\} = \langle a, J^{\intercal}b\rangle$.

Assume that the system in variations (3.22) admits a set of three linearly independent complex solutions $a_k(t) = (\vec{W}_k(t), \vec{Z}_k(t))$ satisfying the skew-orthogonal condition

$$\{a_k(t), a_l(t)\} = 0, \qquad \{a_k(t), a_l^*(t)\} = 2\mathrm{i}\delta_{kl}, \qquad k, l = \overline{1,3}. \tag{3.25}$$

Recall that the six vectors $a_k(t)$ and $a_k^*(t)$, $k = \overline{1,3}$, serve as a symplectic basis in $\mathbb{C}^6_a$, and the three-dimensional plane $r^3(Z(t,\mathfrak{C}))$ with the basis $a_k(t)$ constitutes a complex germ on $z = Z(t,\mathfrak{C})$ [40–42].

Setting $N_j = (2\hbar)^{-1/2}$ in formula (3.23), we compare the vectors $a_j^*(t)$ with the 'creation' operators $\hat{a}_j^+(t)$ and the vectors $a_j(t)$ with the 'annihilation' operators $\hat{a}_j(t)$. Then, taking into account (3.24), we obtain for the operators $\hat{a}_j^+(t)$, $\hat{a}_j(t)$ the bosonic commutation relations

$$[\hat{a}_j(t), \hat{a}_k(t)]_- = \left[\hat{a}_j^+(t), \hat{a}_k^+(t)\right]_- = 0, \qquad \left[\hat{a}_j(t), \hat{a}_k^+(t)\right]_- = \delta_{jk}, \qquad j, k = \overline{1,3}. \tag{3.26}$$

Statement 1. *Let $a_j(t) = (\vec{W}_j(t), \vec{Z}_j(t))$ be solutions of the problem (3.22), (3.25). Then the function $\Phi_0(\vec{x}, t, \mathfrak{C}) = |0, t, \mathfrak{C}\rangle$ (3.17) is a 'vacuum' trajectory-coherent state*

$$\hat{a}_j(t, \mathfrak{C})|0, t, \mathfrak{C}\rangle = 0, \qquad j = \overline{1,3}, \tag{3.27}$$

for the associated linear Schrödinger equation (3.16).

Proof. Applying the 'annihilation' operator $\hat{a}_j(t)$ to the function $|0, t\rangle$, we find that

$$\hat{a}_j|0, t\rangle = |0, t\rangle[\langle\vec{Z}_j(t), Q(t)\Delta\vec{x}\rangle - \langle\vec{W}_j(t), \Delta\vec{x}\rangle].$$

It follows immediately that (3.27) is true as, according to the definition and properties of the matrix $Q(t)$, we have

$$Q(t)\vec{Z}_j(t) = B(t)C^{-1}(t)\vec{Z}_j(t) = \vec{W}_j(t).$$

[5] For simplicity we omit $\hbar$ and $\mathfrak{C}$ in designations of operators and functions.

Let us define now a countable set of states $|\nu, t, \mathfrak{C}\rangle$ (exact solution of equation (3.16)) as a result of the action of the 'creation' operators on the 'vacuum' state $|0, t, \mathfrak{C}\rangle$ (3.17),

$$\Phi_\nu(\vec{x}, t, \mathfrak{C}) = |\nu, t, \mathfrak{C}\rangle = \frac{1}{\sqrt{\nu!}}(\hat{a}^+(t, \mathfrak{C}))^\nu |0, t, \mathfrak{C}\rangle$$
$$= \prod_{k=1}^{3} \frac{1}{\sqrt{\nu_k!}}\left(\hat{a}_k^+(t, \mathfrak{C})\right)^{\nu_k} |0, t, \mathfrak{C}\rangle, \qquad \nu = (\nu_1, \nu_2, \nu_3). \tag{3.28}$$

Note that, in view of the explicit form of the 'vacuum' state (3.17) and symmetry operators (3.23) for the functions $\Phi_\nu(\vec{x}, t)$ (3.28) constituting the Fock basis, the following representation is true:

$$\Phi_\nu(\vec{x}, t) = N_\nu \Phi_0(\vec{x}, t) \mathrm{He}_\nu(\vec{\zeta}(R, t), t). \tag{3.29}$$

Here, $\mathrm{He}_\nu(\vec{\zeta}(R, t), t)$ are many-dimensional Hermite polynomials, which are set by the matrix $W(t)$ (see [43] for details),

$$\mathrm{He}_\nu(\vec{\zeta}(R, t), t) = (-1)^{|\nu|}\left(\frac{\partial}{\partial \vec{\zeta}} - 2W(t)\vec{\zeta}\right)^\nu \cdot 1, \qquad N_\nu = \left(\frac{1}{\sqrt{2}}\right)^{|\nu|} \frac{1}{\sqrt{\nu!}}; \tag{3.30}$$

$$\vec{\zeta}(R, t) = \frac{-\mathrm{i}}{\sqrt{\hbar}}(C^*(t))^{-1}\Delta\vec{x}, \qquad W(t) = C^+(t)(C^{-1}(t))^\intercal. \tag{3.31}$$

The functions $\Phi_\nu(x, t, \mathfrak{C})$ are solutions of equation (3.13) provided that $\mathfrak{C}$ are chosen, so that the solutions $\mathfrak{g}(t, \mathfrak{C})$ of the Hamilton–Ehrenfest system (3.4) coincide with the corresponding mean values $\mathfrak{g}_{\Phi_\nu}(t, \hbar, \mathfrak{C})$ for the states (3.28) at $t = 0$ (see [44]).

Let us designate this set of parameters by $\overline{\mathfrak{C}}_\nu$; then we have

$$\Psi_\nu(x, t, \hbar) = \Phi_\nu(x, t, \overline{\mathfrak{C}}_\nu). \tag{3.32}$$

The subscript ν in $\overline{\mathfrak{C}}_\nu$ indicates that for each function $\Psi_\nu(x, t, \hbar)$ there exists its own set of parameters $\overline{\mathfrak{C}}_\nu$.

The construction of solutions (3.32) uses solutions of two auxiliary ordinary differential systems: the Hamilton–Ehrenfest system (3.4) and the system in variations (3.22).

For an arbitrary time dependence of the coefficients $R(t)$, the solutions of these systems are unknown. However, if the system parameters depend on time adiabatically, we can seek a solution of the Hamilton–Ehrenfest system and system in variations in the form of a power series in the adiabatic parameter, for which $1/T$ is taken. These solutions allow one to construct the leading term of the asymptotics in the parameter $1/T$ of equation (1.1). For the solutions of this type, the Berry phase can be found in an explicit form. □

4. The spectral problem

Consider the spectral problem (2.2) to state the Cauchy problem (1.1), (2.1) and find the dynamic phase by (2.7).

The solution of the spectral problem can be obtained from the non-stationary Hartree-type equation (1.1), where the operator $\widehat{\mathcal{H}}_x(R(t), \Psi(t))$ is replaced by $\widehat{\mathcal{H}}_x(R, \Psi(t))$ with $R = \mathrm{const}$. The solutions of equation (1.1), which have the form

$$\Psi(\vec{x}, t) = \exp\left\{-\frac{\mathrm{i}}{\hbar}E_\nu(R)t\right\}\psi_\nu(\vec{x}, R), \tag{4.1}$$

give a solution of the spectral problem (2.2). Here $\psi_\nu(\vec{x}, R)$ and $E_\nu(R)$ are the eigenfunctions and the eigenvalues of the instantaneous Hamiltonian $\widehat{\mathcal{H}}_x(R, \psi_\nu(R))$, respectively.

The spectral problem is related to the stationary solutions ($\dot{\mathfrak{g}}(t, \mathfrak{C}) = 0$) of the Hamilton–Ehrenfest system (3.4) written for the stationary nonlinear Hamiltonian $\widehat{\mathcal{H}}_\varkappa(R, \psi)$. The solutions of (3.4) determine a stationary point $\mathfrak{g}(R, \mathfrak{C}_s)$ in the corresponding extended phase space. Here $\mathfrak{C}_s$ designates a subset of constants separated from the set $\mathfrak{C}$ by the condition of stationarity of the solutions of the Hamilton–Ehrenfest system (see [22]). From

$$J\mathcal{H}_z(\mathfrak{g}, R) = 0,$$

it follows that $\vec{P}(R, \mathfrak{C}_s) = 0$, $\vec{X}(R, \mathfrak{C}_s) = 0$.

The solutions of the Hamilton–Ehrenfest system for the second-order moments are obtained from solutions of the system in variations according to (3.10).

The linearly independent solutions of equation (3.22), normalized by the skew-orthogonality condition (3.25), can be written as

$$a_\eta(t) = \mathrm{e}^{\mathrm{i}\Omega_\eta t} f_\eta(R), \qquad a_3(t) = \mathrm{e}^{\mathrm{i}\Omega_3 t} f_3(R); \tag{4.2}$$

$$f_\eta(R) = \begin{pmatrix} \frac{\sqrt{m}(\mathrm{i}\rho+\omega_a)}{\sqrt{2\omega_a}}(\mathrm{i}\vec{e}_\varphi + (-1)^\eta \vec{e}_\theta) \\ \frac{1}{\sqrt{2m\omega_a}}(\vec{e}_\varphi - i(-1)^\eta \vec{e}_\theta) \end{pmatrix}, \qquad f_3(R) = \begin{pmatrix} -\frac{\sqrt{m}(\rho-\mathrm{i}\Omega_3)}{\sqrt{\Omega_3}}\vec{e}_n \\ \frac{1}{\sqrt{m\Omega_3}}\vec{e}_n \end{pmatrix}. \tag{4.3}$$

Here we have used the notation

$$\begin{cases} \vec{e}_n = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta), \\ \vec{e}_\varphi = (\sin\varphi, -\cos\varphi, 0), \\ \vec{e}_\theta = (\cos\varphi \cos\theta, \sin\varphi \cos\theta, -\sin\theta), \end{cases} \tag{4.4}$$

$$\Omega_\eta(R) = (-1)^\eta \frac{\omega_c(R)}{2} + \omega_a(R); \qquad \Omega_3(R) = \sqrt{\frac{\tilde{k}}{m} - \rho^2}, \tag{4.5}$$

where

$$\eta = 1, 2; \qquad \omega_c(R) = \frac{eH}{mc}; \qquad \omega_a(R) = \sqrt{\frac{e^2H^2}{4m^2c^2} + \frac{\tilde{k}}{m} - \rho^2}. \tag{4.6}$$

The unit vector $\vec{e}_n$ specifies the direction of the magnetic field, and the set of vectors $\{\vec{e}_\varphi, \vec{e}_\theta, \vec{e}_n\}$ constitutes an orthonormal basis in $\mathbb{R}^3$. We suppose that the frequencies $\Omega_1, \Omega_2, \Omega_3$ do not satisfy the resonance relation $l_1\Omega_1 + l_2\Omega_2 + l_3\Omega_3 = 0$, where l_1, l_2, l_3 are integers.

The stationary solution $\Delta_2(R, \mathfrak{C}_s)$ can be shown to exist if the solutions of the system in variations can be represented in the form (4.2) (see [45]).

Note that, according to (4.2), the matrices $B(t)$ and $C(t)$ can be written as

$$B(t) = \tilde{B}(R)\Lambda(t) = G\tilde{B}_0(R)\Lambda(t), \qquad C(t) = \tilde{C}(R)\Lambda(t) = G\tilde{C}_0(R)\Lambda(t), \tag{4.7}$$

where

$$\tilde{B}_0(R) = \begin{pmatrix} -\frac{\sqrt{m}(\rho-\mathrm{i}\omega_a)}{\sqrt{2\omega_a}} & -\frac{\sqrt{m}(\rho-\mathrm{i}\omega_a)}{\sqrt{2\omega_a}} & 0 \\ -\frac{\sqrt{m}(\mathrm{i}\rho+\omega_a)}{\sqrt{2\omega_a}} & \frac{\sqrt{m}(\mathrm{i}\rho+\omega_a)}{\sqrt{2\omega_a}} & 0 \\ 0 & 0 & -\frac{\sqrt{m}(\rho-\mathrm{i}\Omega_3)}{\sqrt{\Omega_3}} \end{pmatrix}, \tag{4.8}$$

$$\tilde{C}_0(R) = \begin{pmatrix} \frac{1}{\sqrt{2m\omega_a}} & \frac{1}{\sqrt{2m\omega_a}} & 0 \\ \frac{\mathrm{i}}{\sqrt{2m\omega_a}} & \frac{-\mathrm{i}}{\sqrt{2m\omega_a}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{m\Omega_3}} \end{pmatrix}, \tag{4.9}$$

$$\Lambda(t) = \mathrm{diag}\{\exp(\mathrm{i}\Omega_1(R)t), \exp(\mathrm{i}\Omega_2(R)t), \exp(\mathrm{i}\Omega_3(R)t)\}, \tag{4.10}$$

$$G = (\vec{e}_\varphi, \vec{e}_\theta, \vec{e}_n) = \begin{pmatrix} \sin\varphi & \cos\varphi\cos\theta & \cos\varphi\sin\theta \\ -\cos\varphi & \sin\varphi\cos\theta & \sin\varphi\sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}. \tag{4.11}$$

It is easy to verify that the matrix G is orthogonal.

In view of the relations

$$S(t, \mathfrak{C}_s) = -\frac{\tilde{\varkappa}}{2}\sum_{k=1}^{3} c\sigma_{x_k x_k}(R, \mathfrak{C}_s)t, \tag{4.12}$$

$$\det C(t) = \frac{-\mathrm{i}}{m^{3/2}\Omega_3^{1/2}\omega_a}\exp\{\mathrm{i}(\Omega_1 + \Omega_2 + \Omega_3)t\}, \tag{4.13}$$

$$Q(R) = GQ_0(R)G^{\intercal}, \qquad Q_0(R) = \mathrm{diag}\{m(\mathrm{i}\omega_a - \rho), m(\mathrm{i}\omega_a - \rho), m(\mathrm{i}\Omega_3 - \rho)\}, \tag{4.14}$$

$$N_\hbar(\det C(0))^{1/2} = (\pi\hbar)^{-3/4}, \tag{4.15}$$

we obtain from (3.17) the vacuum solution of the associated linear Schrödinger equation

$$\begin{aligned}\Phi_0(\vec{x}, t, \mathfrak{C}_s) = {} & \frac{\sqrt{\mathrm{i}}m^{3/4}\Omega_3^{1/4}\omega_a^{1/2}}{(\pi\hbar)^{3/4}} \\ & \times \exp\left\{\frac{\mathrm{i}}{\hbar}\left(-\frac{\tilde{\varkappa}}{2}\sum_{k=1}^{3} c\sigma_{x_k x_k}(R, \mathfrak{C}_s)t - \frac{\hbar}{2}\sum_{k=1}^{3}\Omega_k t + \frac{1}{2}\langle\vec{\chi}, Q_0(R)\vec{\chi}\rangle\right)\right\} \\ = {} & \exp\left\{-\frac{\mathrm{i}}{\hbar}\left(\frac{\tilde{\varkappa}}{2}\sum_{k=1}^{3} c\sigma_{x_k x_k}(R, \mathfrak{C}_s)t + \frac{\hbar}{2}\sum_{k=1}^{3}\Omega_k t\right)\right\}\phi_0(\vec{x}, R, \mathfrak{C}_s).\end{aligned} \tag{4.16}$$

Here $\vec{\chi} = G^{\intercal}\vec{x}$. In view of (3.29) and (4.7), we have

$$\begin{aligned}\Phi_\nu(\vec{x}, t, \mathfrak{C}_s) = {} & \left(\frac{1}{\sqrt{2}}\right)^{|\nu|}\frac{1}{\sqrt{\nu!}}\exp\left\{-\frac{\mathrm{i}}{\hbar}\left(\frac{\tilde{\varkappa}}{2}\sum_{k=1}^{3} c\sigma_{x_k x_k}(R, \mathfrak{C}_s)t + \sum_{k=1}^{3}\hbar\Omega_k\left(\nu_k + \frac{1}{2}\right)t\right)\right\} \\ & \times \phi_0(\vec{x}, R, \mathfrak{C}_s)H_\nu(\vec{\xi}(R)),\end{aligned} \tag{4.17}$$

where $H_\nu(\vec{\xi}(R))$ are Hermite polynomials, which are set by the matrix $\widetilde{W}(R)$. Here

$$\vec{\xi}(R) = \frac{-\mathrm{i}}{\sqrt{\hbar}}(\widetilde{C}^*(R))^{-1}\Delta\vec{x} = -\mathrm{i}\sqrt{\frac{m}{2\hbar}}\begin{pmatrix} \sqrt{\omega_a} & \mathrm{i}\sqrt{\omega_a} & 0 \\ \sqrt{\omega_a} & -\mathrm{i}\sqrt{\omega_a} & 0 \\ 0 & 0 & \sqrt{2\Omega_3} \end{pmatrix}\vec{\chi}, \tag{4.18}$$

$$\widetilde{W}(R) = -\widetilde{C}_0^{+}(R)(\widetilde{C}_0^{-1}(R))^{\intercal} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.19}$$

For the considered Hermite polynomials, the following relation results from (4.7):

$$\mathrm{He}_\nu(\vec{\zeta}(R, t), t) = \exp\left[-\mathrm{i}\sum_{k=1}^{3}\Omega_k\nu_k t\right]H_\nu(\vec{\xi}(R), R). \tag{4.20}$$

Evidently, to solve the spectral problem under study, it suffices to know only the sub-matrix of the coordinate variances instead of finding the complete matrix of the second moments.

The matrix of the coordinate variances calculated for the states $\Phi_\nu(\vec{x}, t, \mathfrak{C}_s)$ does not depend on $\mathfrak{C}_s$, and the following formula is valid (see, e.g., [39]):

$$\sigma_{xx}^{\Phi_\nu} = \frac{\hbar}{4}[\tilde{C}(R)D_\nu^{-1}\tilde{C}^{+}(R) + \tilde{C}^{*}(R)D_\nu^{-1}\tilde{C}^{\intercal}(R)], \qquad D_\nu^{-1} = \|(2\nu_j + 1)\delta_{kj}\|_{3\times 3}. \tag{4.21}$$

Using (4.21), we find

$$\sum_{k=1}^{3}\sigma_{x_k x_k}(R, \overline{\mathfrak{C}}_\nu) = \left(\nu_1 + \frac{1}{2}\right)\frac{\hbar}{m\omega_a} + \left(\nu_2 + \frac{1}{2}\right)\frac{\hbar}{m\omega_a} + \left(\nu_3 + \frac{1}{2}\right)\frac{\hbar}{m\Omega_3}. \tag{4.22}$$

Then, in view of (3.32), the solution of equation (1.1) is

$$\Psi(\vec{x}, t) = \Phi_\nu(\vec{x}, t, \overline{\mathfrak{C}}_\nu) = \left(\frac{1}{\sqrt{2}}\right)^{|\nu|}\frac{1}{\sqrt{\nu!}} \times \exp\left\{-\mathrm{i}\sum_{k=1}^{3}(\Omega_k + \tilde{\Omega}_k)\left(\nu_k + \frac{1}{2}\right)t\right\} H_\nu(\vec{\xi}(R))\phi_0(\vec{x}, R, \overline{\mathfrak{C}}_\nu), \tag{4.23}$$

$$\tilde{\Omega}_1(R) = \tilde{\Omega}_2(R) = \frac{\tilde{\varkappa}c}{2m\omega_a(R)}, \qquad \tilde{\Omega}_3(R) = \frac{\tilde{\varkappa}c}{2m\Omega_3(R)}. \tag{4.24}$$

Therefore, the eigenfunctions of the Hartree operator (1.2) are

$$\psi_\nu(\vec{x}, R) = \left(\frac{1}{\sqrt{2}}\right)^{|\nu|}\frac{1}{\sqrt{\nu!}}\frac{\sqrt{\mathrm{i}}m^{3/4}\Omega_3^{1/4}\omega_a^{1/2}}{(\pi\hbar)^{3/4}}\exp\left\{\frac{\mathrm{i}}{\hbar}\left(\frac{1}{2}\langle\vec{\chi}, Q_0(R)\vec{\chi}\rangle\right)\right\}H_\nu(\vec{\xi}(R)), \tag{4.25}$$

and the corresponding eigenvalues are given by the expression

$$E_\nu(R) = \hbar\sum_{k=1}^{3}(\Omega_k(R) + \tilde{\Omega}_k(R))\left(\nu_k + \frac{1}{2}\right). \tag{4.26}$$

5. The adiabatic approximation and the Berry phase

Assume that the evolution of a quantum system goes adiabatically. This implies that the Hamiltonian parameters slowly vary with time (see (2.3)) and, along with the 'fast' time t entering the time derivative operator, a 'slow' time s can be introduced which the Hamiltonian parameters ($R(t)$=$R(s)$) depend on. Let the 'fast' and the 'slow' time be related as

$$s = t/T, \tag{5.1}$$

where T is the evolution period of the system.

As mentioned above, to find a solution of equation (1.1) in the adiabatic approximation, it is necessary to solve the Hamilton–Ehrenfest system and the system in variations accurate to the second order in $1/T$.

The Hamilton–Ehrenfest system can be written as

$$\begin{cases}\dfrac{1}{T}z_\Psi' = J\mathcal{H}_z(\mathfrak{g}_\Psi, R(s)),\\ \dfrac{1}{T}\Delta_{2\Psi}' = J\mathcal{H}_{zz}(R(s))\Delta_{2\Psi} - \Delta_{2\Psi}\mathcal{H}_{zz}(R(s))J, \qquad \Delta_2^{\intercal} = \Delta_2,\end{cases} \tag{5.2}$$

where $a' = \mathrm{d}a/\mathrm{d}s$. We seek a solution of this system in the form

$$\vec{Z}(t) = \vec{Z}^{(0)}(s) + \frac{1}{T}\vec{Z}^{(1)}(s) + O\left(\frac{1}{T^2}\right), \qquad \Delta_2(t) = \Delta_2^{(0)}(s) + \frac{1}{T}\Delta_2^{(1)}(s) + O\left(\frac{1}{T^2}\right), \tag{5.3}$$

and obtain

$$\vec{X}^{(0)}(s) = \vec{X}^{(1)}(s) = 0, \qquad \vec{P}^{(0)}(s) = \vec{P}^{(1)}(s) = 0. \tag{5.4}$$

As in the spectral problem, we obtain the solution of the Hamilton–Ehrenfest system for the second-order moments using the solutions of the system in variations.

Making the change of variables by (5.1) to the system in variations (3.22), we obtain

$$\frac{1}{T}a'(t) = J\mathcal{H}_{zz}(s)a(t). \tag{5.5}$$

Let us seek a semiclassical asymptotic solution of the system (5.5) in the form

$$a_k(t) = \mathrm{e}^{\mathrm{i}(T\Phi_k(s)+\phi_k(s))}f_k(t) + O\left(\frac{1}{T^2}\right), \tag{5.6}$$

$$f_k(t) = f_k^{(0)}(s) + \frac{1}{T}f_k^{(1)}(s), \qquad k = \overline{1,3}. \tag{5.7}$$

Substituting (5.6) into (5.5) and equating the coefficients of equal powers of $1/T$, we obtain

$$(J\mathcal{H}_{zz}(s) - \mathrm{i}\Phi_k'(s))f_k^{(0)}(s) = 0,$$
$$(J\mathcal{H}_{zz}(s) - \mathrm{i}\Phi_k'(s))f_k^{(1)}(s) = f_k^{(0)'}(s) + \mathrm{i}\phi_k'(s)f_k^{(0)}(s).$$

Then

$$\Phi_k'(s) = \Omega_k(s) = \Omega_k(R(s)), \qquad k = \overline{1,3} \tag{5.8}$$

$$f_\eta^{(0)}(s) = f_\eta(R(s)), \qquad f_3^{(0)}(s) = f_3(R(s)), \qquad \eta = 1,2, \tag{5.9}$$

where the vectors $f_\eta(R(s))$, $f_3(R(s))$ and the functions $\Omega_k(R(s))$ are determined by (4.3) and (4.5), respectively.

Let us decompose the vectors $f_k^{(1)}(s)$ in the basis vectors $f_k^{(0)}(s)$ and $f_k^{(0)*}(s)$:

$$f_k^{(1)}(s) = \sum_{m=1}^{3}\alpha_{km}(s)f_m^{(0)}(s) + \beta_{km}(s)f_m^{(0)*}(s). \tag{5.10}$$

Then we obtain

$$\phi_k'(s) = \frac{1}{2}\left\{f_k^{(0)'}(s), f_k^{(0)*}(s)\right\}, \tag{5.11}$$

$$\alpha_{km}(s) = \frac{\left\{f_k^{(0)'}(s), f_m^{(0)*}(s)\right\}}{2(\Omega_k(s) - \Omega_m(s))}, \qquad \beta_{km}(s) = \frac{\left\{f_m^{(0)}(s), f_k^{(0)'}(s)\right\}}{2(\Omega_m(s) + \Omega_k(s))}. \tag{5.12}$$

Note that

$$\alpha_{km}(s) = -\alpha_{mk}^*(s), \qquad \beta_{km}(s) = \beta_{mk}(s) \tag{5.13}$$

or, in a matrix form,

$$\mathbf{A}(s) = -\mathbf{A}^+(s), \qquad \mathbf{B}(s) = \mathbf{B}^\intercal(\mathbf{s}), \tag{5.14}$$

where $\mathbf{A}(s) = \|\alpha_{km}(s)\|$ and $\mathbf{B}(s) = \|\beta_{km}(s)\|$. By analogy with (4.7), we write

$$B(t) = \left(\tilde{B}^{(0)}(s) + \frac{1}{T}\tilde{B}^{(1)}(s)\right)\Lambda^{(0)}(s,T) + O\left(\frac{1}{T^2}\right), \tag{5.15}$$

$$C(t) = \left(\tilde{C}^{(0)}(s) + \frac{1}{T}\tilde{C}^{(1)}(s)\right)\Lambda^{(0)}(s,T) + O\left(\frac{1}{T^2}\right), \tag{5.16}$$

$$\Lambda^{(0)}(s,T) = \operatorname{diag}\{T\Phi_1(s) + \phi_1(s),\, T\Phi_2(s) + \phi_2(s),\, T\Phi_3(s) + \phi_3(s)\}. \quad (5.17)$$

Here

$$\tilde{B}^{(0)}(s)) = \tilde{B}(R(s)), \qquad \tilde{C}^{(0)}(s) = \tilde{C}(R(s)),$$
$$\tilde{B}^{(1)}(s) = \tilde{B}^{(0)}(s)\mathbf{A}^{\intercal}(s) + \tilde{B}^{(0)*}(s)\mathbf{B}^{\intercal}(s),$$
$$\tilde{C}^{(1)}(s) = \tilde{C}^{(0)}(s)\mathbf{A}^{\intercal}(s) + \tilde{C}^{(0)*}(s)\mathbf{B}^{\intercal}(s).$$

The matrices $\tilde{B}^{(0)}(s)$ and $\tilde{C}^{(0)}(s)$ are determined by the vectors (5.9). Hereinafter we omit the argument s or t if this does not lead to confusion.

Note that

$$\phi_1' = \frac{-(m\rho)'}{2m\omega_a} + \frac{1}{2}(\langle\vec{e}_\varphi', \vec{e}_\theta\rangle - \langle\vec{e}_\theta', \vec{e}_\varphi\rangle), \qquad \phi_2' = \frac{-(m\rho)'}{2m\omega_a} + \frac{1}{2}(\langle\vec{e}_\theta', \vec{e}_\varphi\rangle - \langle\vec{e}_\varphi', \vec{e}_\theta\rangle),$$
$$\phi_3' = \frac{-(m\rho)'}{2m\Omega_3}; \qquad Q(t) = Q^{(0)}(s) + O\left(\frac{1}{T}\right), \qquad Q^{(0)}(s) = Q(R(s)).$$

For the matrices $\mathbf{A}$ and $\mathbf{B}$ we obtain

$$\mathbf{A} = \begin{pmatrix} \alpha_{11} & 0 & \frac{\gamma}{2(\Omega_1-\Omega_3)}\langle\vec{e}_n', (-\mathrm{i}\vec{e}_\varphi + \vec{e}_\theta)\rangle \\ 0 & \alpha_{22} & \frac{\gamma}{2(\Omega_2-\Omega_3)}\langle\vec{e}_n', (-\mathrm{i}\vec{e}_\varphi - \vec{e}_\theta)\rangle \\ \frac{\gamma}{2(\Omega_3-\Omega_1)}\langle\vec{e}_n', (\mathrm{i}\vec{e}_\varphi + \vec{e}_\theta)\rangle & \frac{\gamma}{2(\Omega_3-\Omega_2)}\langle\vec{e}_n', (\mathrm{i}\vec{e}_\varphi - \vec{e}_\theta)\rangle & \alpha_{33} \end{pmatrix};$$

$$\mathbf{B} = \begin{pmatrix} 0 & \frac{(m\rho-\mathrm{i}m\omega_a)'}{4m\omega_a^2} & \frac{\tilde{\gamma}}{2(\Omega_3+\Omega_1)}\langle\vec{e}_n', (-\mathrm{i}\vec{e}_\varphi + \vec{e}_\theta)\rangle \\ \frac{(m\rho-\mathrm{i}m\omega_a)'}{4m\omega_a^2} & 0 & \frac{\tilde{\gamma}}{2(\Omega_3+\Omega_2)}\langle\vec{e}_n', (-\mathrm{i}\vec{e}_\varphi - \vec{e}_\theta)\rangle \\ \frac{\tilde{\gamma}}{2(\Omega_3+\Omega_1)}\langle\vec{e}_n', (-\mathrm{i}\vec{e}_\varphi + \vec{e}_\theta)\rangle & \frac{\tilde{\gamma}}{2(\Omega_3+\Omega_2)}\langle\vec{e}_n', (-\mathrm{i}\vec{e}_\varphi - \vec{e}_\theta)\rangle & \frac{(m\rho-\mathrm{i}m\Omega_3)'}{4m\Omega_3^2} \end{pmatrix}.$$

Here

$$\gamma = \left[\sqrt{\frac{\Omega_3}{2\omega_a}} + \sqrt{\frac{\omega_a}{2\Omega_3}}\right], \qquad \tilde{\gamma} = \left[\sqrt{\frac{\Omega_3}{2\omega_a}} - \sqrt{\frac{\omega_a}{2\Omega_3}}\right]; \quad (5.18)$$

the matrix elements α_{11}, α_{22} and α_{33} are imaginary, and they are determined from the next-order approximation. We do not give them in an explicit form as these functions do not contribute to the leading term of the asymptotic expansion.

To construct the solutions of the nonlinear equation (1.1), we need the matrix of the coordinate variances for the states $\Phi_\nu(\vec{x}, t, \mathfrak{C})$ (3.28)

$$\sigma_{xx}^{\Phi_\nu}(t) = \frac{\hbar}{4}\big[C(t)D_\nu^{-1}C^{+}(t) + C^{*}(t)D_\nu^{-1}C^{T}(t)\big], \quad (5.19)$$

and it suffices to know this matrix accurate to $O(1/T^2)$:

$$\sigma_{xx}^{\Phi_\nu}(t) = \sigma_{xx}^{(0)}(s) + \frac{1}{T}\sigma_{xx}^{(1)}(s) + O\left(\frac{1}{T^2}\right). \quad (5.20)$$

Here

$$\sigma_{xx}^{(0)} = \frac{\hbar}{4}\big[\tilde{C}^{(0)}D_\nu^{-1}\tilde{C}^{(0)+} + \tilde{C}^{(0)*}D_\nu^{-1}\tilde{C}^{(0)\intercal}\big], \quad (5.21)$$

$$\begin{aligned}\sigma_{xx}^{(1)} &= \frac{\hbar}{4}\big[\tilde{C}^{(0)}D_\nu^{-1}\tilde{C}^{(1)+} + \tilde{C}^{(1)}D_\nu^{-1}\tilde{C}^{(0)+} + \tilde{C}^{(0)*}D_\nu^{-1}\tilde{C}^{(1)T} + \tilde{C}^{(1)*}D_\nu^{-1}\tilde{C}^{(0)T}\big] \\ &= \frac{\hbar}{4}\big[\tilde{C}^{(0)}\big\{\mathbf{A}^{\intercal}D_\nu^{-1} - D_\nu^{-1}\mathbf{A}^{\intercal}\big\}\tilde{C}^{(0)+} + \tilde{C}^{(0)}\big\{D_\nu^{-1}\mathbf{B}^{*} + \mathbf{B}^{*}D_\nu^{-1}\big\}\tilde{C}^{(0)\intercal} \\ &\quad + \tilde{C}^{(0)*}\big\{D_\nu^{-1}\mathbf{A} - \mathbf{A}D_\nu^{-1}\big\}\tilde{C}^{(0)\intercal} + \tilde{C}^{(0)*}\big\{D_\nu^{-1}\mathbf{B} + \mathbf{B}D_\nu^{-1}\big\}\tilde{C}^{(0)+}\big]. \end{aligned} \quad (5.22)$$

Similar to (4.22) we obtain

$$\sum_{k=1}^{3}\sigma_{x_kx_k}(t,\overline{\mathfrak{C}}_\nu)=\left(\nu_1+\frac{1}{2}\right)\left[\frac{\hbar}{m\omega_a}+\frac{1}{T}\frac{\hbar(m\rho)'}{2m^2\omega_a^3}\right]$$
$$+\left(\nu_2+\frac{1}{2}\right)\left[\frac{\hbar}{m\omega_a}+\frac{1}{T}\frac{\hbar(m\rho)'}{2m^2\omega_a^3}\right]+\left(\nu_3+\frac{1}{2}\right)\left[\frac{\hbar}{m\Omega_3}+\frac{1}{T}\frac{\hbar(m\rho)'}{2m^2\Omega_3^3}\right]+O\left(\frac{1}{T^2}\right).$$

In view of this, a solution of the Cauchy problem (1.1), (2.1) can be represented in the form

$$\Psi(\vec{x},t)=\Psi_\nu^{(0)}(\vec{x},t)+O\left(\frac{1}{T}\right), \tag{5.23}$$

where

$$\Psi_\nu^{(0)}(\vec{x},t)=\exp\left\{-\frac{\mathrm{i}}{\hbar}T\int_0^s E_\nu(\tau)\,\mathrm{d}\tau+\mathrm{i}\gamma_\nu(s)\right\}\psi_\nu(\vec{x},R(s)). \tag{5.24}$$

The function (5.24) is a solution of equation (1.1) in the adiabatic approximation. Here the functions $\psi_\nu(\vec{x},R(s))$ are instantaneous eigenfunctions of $\hat{\mathcal{H}}_\varkappa(R(s),\psi_\nu(R(s)))$. The quantities

$$E_\nu(s)=\hbar\sum_{j=1}^{3}(\Omega_j(s)+\tilde{\Omega}_j(s))\left(\nu_j+\frac{1}{2}\right) \tag{5.25}$$

are eigenvalues of the instantaneous Hartree-type Hamiltonian $\hat{\mathcal{H}}_\varkappa(R(s),\psi_\nu(R(s)))$ and $\gamma_\nu(s)$ is

$$\gamma_\nu(s)=-\int_0^s\sum_{j=1}^{3}\left[\phi_j'(\tau)\left(\nu_j+\frac{1}{2}\right)+\frac{\tilde{\varkappa}c(\tau)}{2\hbar}\sum_{j=1}^{3}\sigma_{x_jx_j}^{(1)}(\tau)\right]\mathrm{d}\tau$$
$$=\sum_{j=1}^{2}\left(\nu_j+\frac{1}{2}\right)\int_0^s\left[1-\frac{\tilde{\varkappa}c(\tau)}{2m(\tau)\omega_a^2(\tau)}\right]\frac{(m(\tau)\rho(\tau))'}{2m(\tau)\omega_a(\tau)}\,\mathrm{d}\tau$$
$$+\left(\nu_3+\frac{1}{2}\right)\int_0^s\left[1-\frac{\tilde{\varkappa}c(\tau)}{2m(\tau)\Omega_3^2(\tau)}\right]\frac{(m(\tau)\rho(\tau))'}{2m(\tau)\Omega_3(\tau)}\,\mathrm{d}\tau$$
$$+\frac{1}{2}\left(\nu_2-\nu_1\right)\int_0^s\langle e_\varphi'(\tau),e_\theta(\tau)\rangle-\langle e_\theta'(\tau),e_\varphi(\tau)\rangle\,\mathrm{d}\tau. \tag{5.26}$$

The functions $\Omega_j(s)$, $\omega_a(s)$ and $\tilde{\Omega}_j(s)$ are defined by (4.5), (4.6) and (4.24), respectively. The evolution of the function (5.24) in the period T is described by

$$\Psi_\nu^{(0)}(\vec{x},T)=\exp\left\{-\frac{\mathrm{i}}{\hbar}T\int_0^1 E_\nu(s)ds+\mathrm{i}\gamma_\nu(T)\right\}\Psi_\nu^{(0)}(\vec{x},0). \tag{5.27}$$

We now use (2.6) and (2.7) to determine the dynamic phase

$$\delta_\nu(T)=T\int_0^1\hbar\sum_{j=1}^{3}\left(\nu_j+\frac{1}{2}\right)[\Omega_j(s)+\tilde{\Omega}_j(s)]\,\mathrm{d}s \tag{5.28}$$

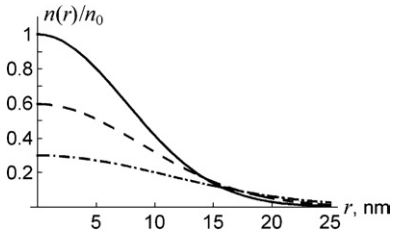

Figure 1. Distribution density $n(r)$ versus radius r for different relations between k and a for the ground state ($\nu = (0, 0, 0)$). The density $n(r)$ is related to n_0, where $n_0 = n(0)|_{\tilde{x}=0}$. The radius r is given in nanometers.

and the Berry phase

$$\gamma_\nu(T) = \sum_{j=1}^{2}\left(\nu_j + \frac{1}{2}\right)\oint_C\left[1 - \frac{\tilde{\varkappa}c}{2m\omega_a^2}\right]\frac{1}{2\omega_a}\left(\frac{\rho}{m}\,\mathrm{d}m + \mathrm{d}\rho\right)$$
$$+\left(\nu_3 + \frac{1}{2}\right)\oint_C\left[1 - \frac{\tilde{\varkappa}c}{2m\Omega_3^2}\right]\frac{1}{2\Omega_3}\left(\frac{\rho}{m}\,\mathrm{d}m + \mathrm{d}\rho\right)$$
$$+(\nu_2 - \nu_1)\oint_C \frac{H_3}{H\left(H_1^2 + H_2^2\right)}[H_1\,\mathrm{d}H_2 - H_2\,\mathrm{d}H_1]. \tag{5.29}$$

Here H_1, H_2 and H_3 are components of the magnetic field, $H = \sqrt{H_1^2 + H_2^2 + H_3^2}$; C is a contour in the parameter space.

By analogy with the theory of Bose–Einstein condensates (BEC) [20], we assume that the solutions of equation (5.24) are normalized by N, where N is the number of the system particles. Then the function $n(\vec{x}, t) = |\Psi(\vec{x}, t)|^2$ describes the density of distribution of the system particles. Consider the effect of nonlinearity on the distribution density $n(\vec{x}, t)$ for the spherically symmetric case $\vec{H} = 0, \rho(t) = 0$. Let us choose the values of the system parameters corresponding to the Paul trap for Be ions: $m = 1.46 \times 10^{-26}$kg, $\omega = \sqrt{k/m} = 2\pi \times 10$ MHz [30]. The dependence of the distribution density $n(r)$ on the radius r is presented in figure 1 for different relations between k and $\tilde{\varkappa}a$ in the ground state ($\nu = (0, 0, 0)$). Three cases are considered in the figure: $\tilde{\varkappa}a = 0$ (solid line), $\tilde{\varkappa}a = -0.5k$ (dashed line) and $\tilde{\varkappa}a = -0.8k$ (dash-dotted line). Note that the negative values of $\tilde{\varkappa}a$ refer to repelling particles. It can be seen that as the mutual repulsion increases, the size of the wave packet also increases, and, accordingly, the maximum of the density decreases. Such a behavior of the distribution density is observed for the BEC of alkali metals with repulsively interacting particles [20]. For the chosen system parameters, a critical value $\tilde{\varkappa}a = -k$ arises when bound states cannot exist because the external field is not able to confine repelling ions.

To illustrate the effect of the magnetic field on the system, consider the behavior of the distribution density in an alternating magnetic field of the form $H_1(t) = H_0\cos(\tilde{\omega}t)$, $H_2(t) = H_0\sin(\tilde{\omega}t)$, $H_3 = H_0$. Here $\tilde{\omega}$ is the frequency of the magnetic field, and H_0 characterizes the intensity of the magnetic field. The value of H_0 is taken such that $\omega_c = 10\omega$. The frequency $\tilde{\omega}$ is equal to $\omega/100$, providing the adiabatic evolution of the system parameters. Figure 2 shows the distribution density $n(\vec{x}, t)$ in the plane (x_1, x_3) at different points in time. A change of the magnetic-field direction causes a redistribution of the density. Such a behavior is due to the many-dimensional character of the problem under consideration because these solutions cannot be obtained by a mere multiplication of the one-dimensional solutions.

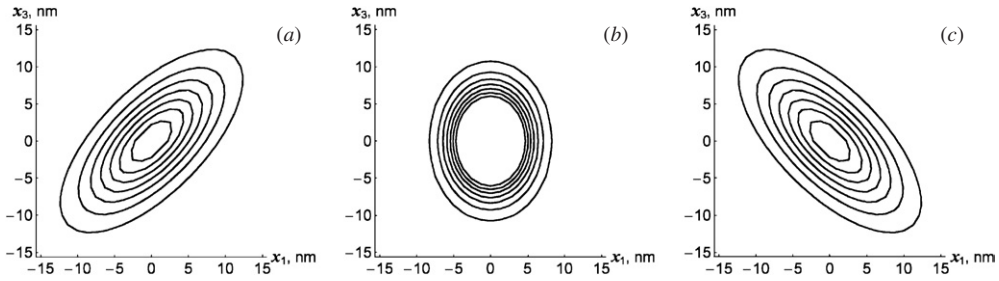

Figure 2. Contour plots of the density $n(\vec{x}, t)$ in the plane (x_1, x_3) at different times within the period $T = 2\pi/\tilde{\omega}$ of the magnetic field evolution: $t_0 = 0$, $t_4 = 2\pi/\tilde{\omega}$ (a), $t_1 = \pi/2\tilde{\omega}$, $t_3 = 3\pi/2\tilde{\omega}$ (b) and $t_2 = \pi/\tilde{\omega}$ (c).

It should also be noted that the distribution density domain contracts with increasing magnetic field H, while increasing ρ results in its expansion.

The Berry phase (5.29) of the nonlinear equation (1.1) differs from that of the linear Schrödinger equation ($\tilde{\varkappa} = 0$) by the parameter $\tilde{k}$ instead of k and by an additional summand proportional to $\tilde{\varkappa}$.

Let $a = c$ in (1.4). Then for $(\Omega_{\rm nl}/\Omega_0)^2 \ll 1$ we obtain

$$\gamma_\nu(T) = \sum_{j=1}^{2}\left(\nu_j + \frac{1}{2}\right)\oint_C\left[1 - \xi\left(\frac{\Omega_{\rm nl}}{\omega_0}\right)^2\right]\frac{1}{2\omega_0}\left(\frac{\rho}{m}\,\mathrm{d}m + \mathrm{d}\rho\right)$$
$$+\left(\nu_3 + \frac{1}{2}\right)\oint_C\left[1 - \xi\left(\frac{\Omega_{\rm nl}}{\Omega_0}\right)^2\right]\frac{1}{2\Omega_0}\left(\frac{\rho}{m}\mathrm{d}m + \mathrm{d}\rho\right)$$
$$+(\nu_2 - \nu_1)\oint_C \frac{H_3}{H\left(H_1^2 + H_2^2\right)}[H_1\,\mathrm{d}H_2 - H_2\,\mathrm{d}H_1],$$

$$\Omega_{\rm nl} = \sqrt{\left|\frac{\tilde{\varkappa}a}{m}\right|}, \qquad \xi = \mathrm{sign}\left(\frac{\tilde{\varkappa}a}{m}\right), \qquad \Omega_0 = \Omega_3\big|_{\tilde{\varkappa}=0} = \sqrt{\frac{k}{m} - \rho^2},$$

$$\omega_0 = \omega_a\big|_{\tilde{\varkappa}=0} = \sqrt{\frac{e^2H^2}{4m^2c^2} + \frac{k}{m} - \rho^2}.$$

In the Berry phase theory, the points of the parameter space at which the spectrum of the instantaneous Hamiltonian degenerates play an important part. For example, the Berry phase of a two-level system described by the Pauli Hamiltonian is equal to half the solid angle $\Omega(C)$ that the contour C in the parameter space subtends at a degeneracy point [1].

If the eigenfunctions are real, the parameter space becomes two-dimensional and the solid angle equals 2π when the contour C encircles the degeneracy and vanishes otherwise.

For system (1.1) the spectrum (4.26) of the instantaneous Hamiltonian $\widehat{\mathcal{H}}_\varkappa(R, \Psi(t))$ degenerates as $\vec{H} = 0$. Let us fix the system parameters except for $\vec{H}(t)$. Then the Berry phase (5.29) equals $(\nu_2 - \nu_1)\Omega(\tilde{C})$, where $\Omega(\tilde{C})$ is the solid angle in which the contour $\tilde{C}$ is seen from the point $\vec{H} = 0$ in the three-dimensional space of parameters (H_1, H_2, H_3). If, e.g., $H_2 = 0$ and a plain contour $\tilde{C}$ includes the point $\vec{H} = 0$, then the Berry phase (5.29) of system (1.1) is multiple of 2π in contrast to the two-level system. When the system has several points of degeneration there arise a variety of dependences of the Berry phase on the contour. For such systems, this problem was studied in [46] and the points of degeneracy were investigated in many papers (see for details, e.g., [47] and references therein).

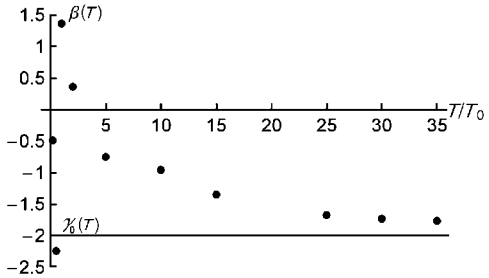

Figure 3. Dependence of $\beta(T)$ versus ratio T/T_0. Here $\gamma_0(T) = -2$ is the value of the Berry phase.

Equation (5.26) can be considered an open path phase (OPP), which can be as important as the Berry phase [48]. In [46], the OPP was studied for different contours in the parameter space and both the Berry phase and the OPP were shown to be proportional to π.

The properties of the OPP, including the invariance with respect to the gauge transformation of the initial function $\psi_n(R) \to \exp[i\mu(R)]\psi_n(R)$, were discussed in [48].

Let us illustrate numerically how the phase of the ground state $\Psi_0(\vec{x}, t)$ of the form (3.32) is related to the Berry phase. To this end consider the function $\lambda(t) = \operatorname{Arg}\sqrt{\frac{\det C(0)}{\det C(t)}} + \frac{1}{\hbar}S(t, \mathfrak{C}_0)$ which determines the time-dependent part of the overall phase of the state $\Psi_0(\vec{x}, t)$. The function $\lambda(t)$ tends to the overall phase $\phi_0(t)$ of the adiabatic wavefunction (2.4) at the adiabatic evolution ($T \to \infty$). Respectively, the function

$$\beta(t) = \lambda(t) + \frac{1}{\hbar}\int_0^t E_0(\tau)\,d\tau \tag{5.30}$$

should tend to the OPP $\gamma_0(t)$ of the form (5.26), and the quantity $\beta(T)$ should tend to the Berry phase $\gamma_0(T)$ (5.29). To show this we calculate the function $\beta(t)$ for different periods T and for the parameters of equation (1.1) which are equal to $m = 1.4, k(t) = \omega^2 m(1+0.4\cos(\tilde{\omega}t))$, $\rho(t) = 0.8\omega\sin(\tilde{\omega}t)$, $\omega_c = 0.5\omega\cos(\tilde{\omega}t)$, $\tilde{\varkappa}a = \tilde{\varkappa}c = -0.2\omega^2 m$ where $\omega = 2\pi \times 10^2$. The adiabatic parameter is $\tilde{\omega}/\omega = T_0/T$ where $T_0 = 2\pi/\omega = 10^{-2}$, $T = 2\pi/\tilde{\omega}$.

Figure 3 shows values $\beta(T)$ for different ratios T/T_0. When the ratio increases, $\beta(T)$ tends to the Berry phase $\gamma_0(T)$.

Figure 4 shows the function $\beta(t)$ for different periods T and the function $\gamma_0(t)$. All curves are plotted in the same figure for convenience (similar to figure 1 of [46]). It is seen that at large enough T the functions $\beta(t)$ and $\gamma_0(t)$ become hardly distinguishable.

Such behavior of $\beta(t)$ and $\beta(T)$ as T increases shows that, indeed, formula (5.29) gives us the Berry phase for the Hartree-type equation (1.1).

The classical analog of the Berry phase is associated with the Hannay angles (see, e.g., [8]). Geometrically, the Hannay angles are similar to the Berry phases. The Berry phase γ_ν of a quantum system and the Hannay angles Θ_i of the corresponding classical system [7] are related by

$$\Theta_i = -\hbar\frac{\partial\gamma_\nu}{\partial I_i} = -\frac{\partial\gamma_\nu}{\partial\nu_i}, \qquad i = \overline{1,3}. \tag{5.31}$$

Here I_i are quantized action variables and ν_i are quantum numbers. The differentiation in (5.31) implies that ν_i is a continuous parameter. In the nonlinear case, it is natural to relate

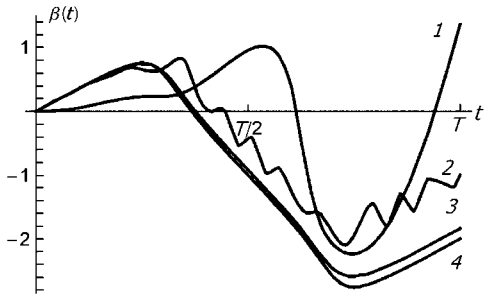

Figure 4. The function $\beta(t)$ of the form (5.30) for different periods T: (1) $T = T_0$, (2) $T = 10T_0$, (3) $T = 50T_0$; the curve (4) shows $\gamma_0(t)$ obtained from (5.26).

the Berry phase (5.29) with an analog of the Hannay angle by formula (5.31) to obtain

$$\Theta_1^{\varkappa} = -\oint\left[1 - \frac{\tilde{\varkappa}c}{2m\omega_a^2}\right]\frac{1}{2\omega_a}\left(\frac{\rho}{m}\mathrm{d}m + \mathrm{d}\rho\right) + \oint_C \frac{H_3}{H\left(H_1^2 + H_2^2\right)}[H_1\,\mathrm{d}H_2 - H_2\,\mathrm{d}H_1],$$

$$\Theta_2^{\varkappa} = -\oint\left[1 - \frac{\tilde{\varkappa}c}{2m\omega_a^2}\right]\frac{1}{2\omega_a}\left(\frac{\rho}{m}\mathrm{d}m + \mathrm{d}\rho\right) - \oint_C \frac{H_3}{H\left(H_1^2 + H_2^2\right)}[H_1\,\mathrm{d}H_2 - H_2\,\mathrm{d}H_1],$$

$$\Theta_3^{\varkappa} = \oint\left[1 - \frac{\tilde{\varkappa}c}{2m\Omega_3^2}\right]\frac{1}{2\Omega_3}\left(\frac{\rho}{m}\mathrm{d}m + \mathrm{d}\rho\right).$$

When $\rho = 0$, the 'Hannay angle' $\Theta_1^{\varkappa}$ is equal to the solid angle $\Omega(\tilde{C})$ at which the curve $\tilde{C}$ can be seen from the origin point in the three-dimensional parameter space (H_1, H_2, H_3).

We have defined the Hannay angle in terms of quantum mechanics since the nonlinear problem requires a special study of the 'classical equations' corresponding to the nonlinear 'quantum' Hartree-type equation. In our consideration, the part of these classical equations is played by the Hamilton–Ehrenfest system (3.4), which has no Hamiltonian form with respect to the standard Poisson bracket.

6. Conclusion

A countable set of solutions has been constructed for the Hartree-type equation (1.2) in the adiabatic approximation and the corresponding Berry phases have been found in an explicit form.

In the linear case ($\tilde{\varkappa} = 0$), the Berry phase is completely determined by the solution of the Hamilton system and the corresponding complex germ [37]. In the nonlinear case, the Hamilton–Ehrenfest system involves the Hamilton system and the complex germ. For the quadratic Hartree-type operator (1.2), the Hamilton–Ehrenfest system (3.4) breaks into the Hamilton system with a self-action (the first equation of (3.4)) and the system in variations (3.11).

It should be noted that, as in the linear case, the leading term of the asymptotics (5.24) in the parameter $1/T$ persists to be an eigenfunction of the nonlinear Hamiltonian (1.2) at every point in time and only gains a phase factor. However, the Berry phase for the nonlinear Hartree-type equation (1.1) cannot be calculated by formula (2.8). This is because the linear superposition principle was essentially used in deriving formula (2.8). The Berry phase is obtained by extracting the dynamic part (by formulae (2.6), (2.7)) from the overall phase incursion in the period T.

There exists a limit as $\tilde{\varkappa} \to 0$ for each expression obtained, and the limits agree with the results of linear quantum theory (see, e.g., [5, 6, 37]).

The geometric Aharonov–Anandan phase [49] is a generalization of the Berry phase for the case of arbitrary cyclic states ($\Psi(T) = \exp[\Phi(T)]\Psi(0)$).

The geometric Aharonov–Anandan phase for a linear Schrödinger equation coincides with the Berry phase in the limiting case of an adiabatically evolving system ($T \to \infty$) [6].

The generalization of the Aharonov–Anandan phase for the cyclic solutions of nonlinear equations with unitary nonlinearity, to which class equation (1.1) also belongs, is given in [50].

For the adiabatic evolution of a system, the limit of the Aharonov–Anandan phase, obviously depends on the type of the nonlinear equation and on the class of functions in which its solutions are sought. The relationship between the Aharonov–Anandan phase as $T \to \infty$ and the Berry phase for the solutions of the nonlinear Hartree-type equation (1.1) in the class of functions (3.12) can be the subject of further research.

From the viewpoint of practical implementation in quantum computations, the non-Abelian Berry phase is important [52, 51] which corresponds to a degenerate spectrum of the Hamiltonian. The method developed in the present work can be easily generalized for non-Abelian phases. Note that, in contrast to the linear case, the occurrence of a non-Abelian phase is related not to the degeneration of the spectrum of the nonlinear operator, but to the degeneration of the spectrum of the corresponding associated linear operator. A comprehensive study of this problem is beyond the scope of this work and requires special consideration.

Acknowledgments

The work was supported in part by President of the Russian Federation (grant no SS-5103.2006.2), DFG, Litvinets F was supported in part by the scholarship of the (nonprofit) Dynasty Foundation.

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